

Locating a public good on a sphere

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Locating a public good on a sphere

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Abstract

It is shown that in a model where agents have single-peaked preferences on the sphere, every Pareto optimal social choice function that is strict or coalitional strategy-proof, is dictatorial.

Journal of Economic Literature Classification D71

Keywords Single-peaked preferences, Pareto optimality, strict strategy-proofness, coalitional strategy-proofness

1 Introduction

It is well known that restricting preferences to be single-peaked may be a means to escape the Gibbard (1973) and Satterthwaite (1975) Theorem, which says that strategy-proof social choice functions on the full preference domain and with range cardinality at least three, are dictatorial. Single-peaked preferences were studied already in Black (1948). Further references include Moulin (1980), Kim and Roush (1981), Border and Jordan (1983), and Peters et al (1992), and this list is not exhaustive. In all those papers, typically, coordinatewise median-like rules turn out to be strategy-proof, besides satisfying other desirable properties. In the present paper, agents have single-peaked preferences on the sphere, i.e., the surface of the three-dimensional unit ball. We show that every Pareto optimal rule that is strict strategy-proof or coalitional strategy-proof must be dictatorial – this in spite of the single-peaked preference assumption. Here, strict strategy-proofness means that if an agent misreports, either he is strictly worse off or the outcome assigned by the rule does not change at all.

It is still an open problem whether mere strategy-proofness and Pareto optimality imply dictatorship: an example at the end of the paper suggests that it does, but we do not have a proof. The example concerns a coordinatewise median rule, which, in many models, is typically not only strategy-proof, but also strict and coalitional strategy-proof.

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We follow similar arguments as in Öztürk et al (2013), who consider single-dipped preferences on the two-dimensional disc. Of course, on the sphere, every single-peaked preference is a single-dipped preference if we take the anti-podal point of the peak as the dip – so the results in this paper also hold for single-dipped preferences.

2 Model and preliminaries

The set of *alternatives* is the unit sphere $A = \{x \in \mathbb{R}^3 : |x| = 1\}$, where $|\cdot|$ denotes Euclidian distance. The set of *agents* is $N = \{1, \dots, n\}$ with $n \geq 2$. Subsets of N are called *coalitions*. Each agent $i \in N$ has a *single-peaked preference* on A , denoted by its *peak* $p(i)$: a point $x \in A$ is weakly preferred to a point $y \in A$ if $|x - p(i)| \leq |y - p(i)|$. Here, distance is measured along the sphere, but since this distance is isomorphic with Euclidian distance within the closed unit ball, we just keep the same notation $|\cdot|$.

A *profile* is a vector $p = (p(1), p(2), \dots, p(n)) \in A^N$. A *social choice function* or *rule* φ assigns to each $p \in A^N$ a collective decision $\varphi(p) \in A$. A point $x \in A$ is *Pareto optimal* for a profile $p \in A^N$ if there is no $x' \in A \setminus \{x\}$ such that $|x' - p(i)| \leq |x - p(i)|$ for all $i \in N$ with at least one strict inequality. Rule φ is *Pareto optimal* if $\varphi(p)$ is Pareto optimal for every $p \in A^N$.

Let $S \subseteq N$. Profiles p and q are *S-deviations* if $p(i) = q(i)$ for all $i \in N \setminus S$. Rule φ is *manipulable by $S \subseteq N$ at $p \in A^N$ via $q \in A^N$* if p, q are *S-deviations* and $|\varphi(p) - p(i)| \geq |\varphi(q) - q(i)|$ for all $i \in S$, with at least one inequality strict; *manipulable by $S \subseteq N$* if there are *S-deviations* $p, q \in A^N$ such that φ is manipulable at p via q ; *coalitional strategy-proof* if it is not manipulable by any $S \subseteq N$; *intermediate strategy-proof* if it is not manipulable by any S at p via q such that $p(i) = p(j)$ for all $i, j \in S$; *strategy-proof* if it is not manipulable by any $\{i\}$, $i \in N$; *strict strategy-proof* if for all $i \in N$ and $\{i\}$ -deviations p and q , either $\varphi(p) = \varphi(q)$ or $|\varphi(p) - p(i)| < |\varphi(q) - p(i)|$; and *intermediate strict strategy-proof* if for all $S \subseteq N$ and *S-deviations* $p, q \in A^N$ with $p(i) = p(j)$ for all $i, j \in S$, either $\varphi(p) = \varphi(q)$ or $|\varphi(p) - p(i)| < |\varphi(q) - p(i)|$ for all $i \in S$.

A coalition $S \subseteq N$ is *decisive for φ* if for every $x \in A$ and every profile $p \in A^N$ with $p(i) = x$ for all $i \in S$ we have $\varphi(p) = x$. If for a particular agent d coalition $\{d\}$ is decisive, then d is a *dictator*. In that case, φ is *dictatorial*.

In this paper we will show for a Pareto optimal rule φ which is strict strategy-proof or coalitional strategy-proof, that the set of decisive coalitions is an ultrafilter. In general, a collection \mathcal{D} of coalitions is an *ultrafilter* if (i) $N \in \mathcal{D}$; (ii) for each $S \subseteq N$ either $S \in \mathcal{D}$ or $N \setminus S \in \mathcal{D}$; and (iii) $S \cap T \in \mathcal{D}$ for all $S, T \in \mathcal{D}$. It is well-known and easy to see that an ultrafilter \mathcal{D} contains a unique singleton $\{d\}$. Hence, if the set of decisive coalitions for a rule φ is an ultrafilter, then φ is dictatorial.¹

We state the following preliminary result.

¹This is a familiar approach, see Kirman and Sondermann (1972) and Hansson (1976).

Lemma 2.1. *Let φ be a Pareto optimal rule that is strict strategy-proof or coalitional strategy-proof. Let \mathcal{D} be the set of decisive coalitions for φ . Then (i) $N \in \mathcal{D}$ and (ii) for each $S \subseteq N$ either $S \in \mathcal{D}$ or $N \setminus S \in \mathcal{D}$.*

Proof. (i) follows from Pareto optimality of φ . For (ii), note that φ is strategy-proof. Then (ii) follows by similar arguments as the corresponding Lemma 2.5 in Öztürk et al (2013). ■

The remainder of the paper will concentrate on proving (iii) for φ . We conclude this section with some spherical terminology, notations, and facts.

A *great circle* G is the intersection of A with a plane through the origin. A great circle G divides A into two disjoint, open *hemispheres*.

Each point $x \in A$ has an *antipodal*, i.e., the point of A with maximal distance to x . For distinct non-antipodal points $x, y \in A$, $[x \frown y]$ denotes the shortest closed arc between x and y , which is a subset of the great circle through x and y . Notations like $[x \frown y)$, $(x \frown y]$, and $(x \frown y)$ are self-explanatory. If x and y are antipodal points then $[x \frown y] = A$. If $x = y$ then $[x \frown y] = \{x\}$.

The *perpendicular bisector* of a closed arc $[x \frown y]$ is the great circle that passes through the midpoint of $[x \frown y]$ and is perpendicular to $[x \frown y]$.

A set $C \subseteq A$ is *convex* if it contains all the arcs $[x \frown y]$ joining any two points $x, y \in C$. The convex hull of a set $X \subseteq A$ is the set $Co(X) = \cap \{C : X \subseteq C, C \text{ convex}\}$.

Remark 2.2. Let G be a great circle and let H be one of the hemispheres induced by G . Let $p \in A^N$ such that $p(i) \in G \cup H$ for all $i \in N$, and $p(i) \in H$ for some $i \in N$. Then $x \in A$ is Pareto optimal for p if and only if $x \in Co(\{p(1), p(2), \dots, p(n)\})$.

3 Strict strategy-proofness and coalitional strategy-proofness

In this section we show that for a Pareto optimal rule the set of decisive coalitions is closed under intersection if the rule is strict strategy-proof or coalitional strategy-proof. We start with considering strict strategy-proofness. In the proof of the following proposition, we use the easily established facts that strict strategy-proofness implies intermediate strategy-proofness and intermediate strict strategy-proofness.

Proposition 3.1. *Let φ be Pareto optimal and strict strategy-proof, and let S and T be decisive coalitions. Then $S \cap T$ is decisive.*

Proof. Contrary to what we wish to prove, suppose that $S \cap T$ is not decisive. Let $X = S \setminus T$, $Y = S \cap T$, and $Z = N \setminus S$. Hence $X \cup Y = S$ and $Y \cup Z = N \setminus (S \setminus T) \supseteq T$ are decisive since S and T are decisive, and $X \cup Z = N \setminus (S \cap T)$ is decisive by Lemma 2.1.

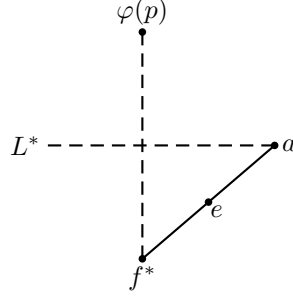


Figure 1: Second part of the proof of Proposition 3.1

Let G be a great circle with equidistant points a , b , and c , and let $p \in A^N$ with $p(i) = a$ for all $i \in X$, $p(i) = b$ for all $i \in Y$, and $p(i) = c$ for all $i \in Z$. We denote $p = (a^X, b^Y, c^Z)$. We first show that $\varphi(p) \notin G$.

Since $X \cup Y$ is decisive, hence $\varphi(b^{X \cup Y}, c^Z) = b$ and $\varphi(a^{X \cup Y}, c^Z) = a$, it follows by strict strategy-proofness that

$$\begin{aligned} |a - \varphi(p)| &< |a - b| \text{ or } \varphi(p) = b \\ \text{and } |b - \varphi(p)| &< |b - a| \text{ or } \varphi(p) = a. \end{aligned} \quad (1)$$

Similarly, since $X \cup Z$ and $Y \cup Z$ are decisive, we derive

$$\begin{aligned} |c - \varphi(p)| &< |c - a| \text{ or } \varphi(p) = a \\ \text{and } |a - \varphi(p)| &< |a - c| \text{ or } \varphi(p) = c \end{aligned} \quad (2)$$

and

$$\begin{aligned} |c - \varphi(p)| &< |c - b| \text{ or } \varphi(p) = b \\ \text{and } |b - \varphi(p)| &< |b - c| \text{ or } \varphi(p) = c. \end{aligned} \quad (3)$$

If $\varphi(p) = a$, then (3) is violated, hence $\varphi(p) \neq a$, and similarly $\varphi(p) \neq b$ and $\varphi(p) \neq c$. If $\varphi(p) \in (a \frown b)$, then (2) is violated, hence $\varphi(p) \notin (a \frown b)$. Similarly, $\varphi(p) \notin (a \frown c)$ and $\varphi(p) \notin (b \frown c)$. Therefore, $\varphi(p) \notin G$.

Let H_1 and H_2 be the two hemispheres separated by G . Assume without loss of generality that $\varphi(p) \in H_1$. For any $x \in A$, we denote the profile (x^X, b^Y, c^Z) by p^x , so $p = p^a$. Consider a sequence of points $(d_k)_{k \in \mathbb{N}} \in H_2$ converging to a . By Remark 2.2, $f^k := \varphi(p^{d_k}) \in H_2 \cup G$ for every $k \in \mathbb{N}$. Since $H_2 \cup G$ is compact we may assume that $(f^k)_{k \in \mathbb{N}}$ converges to some point $f^* \in H_2 \cup G$. Now $f^* \neq a$, otherwise $|f^k - a| < |\varphi(p) - a|$ for large k , so X could manipulate via p^{d_k} at p , contradicting intermediate strategy-proofness.

Let the great circle L^* be the perpendicular bisector of $[\varphi(p) \frown f^*]$. Let (L^*, f^*) and $(L^*, \varphi(p))$ denote the two hemispheres induced by L^* and containing the points f^* and $\varphi(p)$, respectively. If $a \in (L^*, f^*)$, then for large k we have $|a - \varphi(p)| > |a - f^k|$, so that as before X can manipulate at p via p^{d_k} . If $a \in (L^*, \varphi(p))$, hence $|a - \varphi(p)| < |a - f^*|$, then for large k we have $|d_k - \varphi(p)| < |d_k - f^k|$: then X can manipulate at p^{d_k} via p , contradicting again intermediate strategy-proofness. Hence $a \in L^*$.

Let $e \in (f^* \frown a)$ (See Figure 1). Since $|a - \varphi(p)| = |a - f^*|$ we have $\varphi(p^e) \neq f^*$, otherwise intermediate strict strategy-proofness would be violated at p via p^e .

Let $e_k \in (f^k \frown d_k)$, $k \in \mathbb{N}$, such that $(e_k)_{k \in \mathbb{N}}$ converges to e . For every $k \in \mathbb{N}$, by intermediate strategy-proofness, $|\varphi(p^{e_k}) - e^k| \leq |\varphi(p^{d_k}) - e^k|$, otherwise X could manipulate at p^{e_k} via p^{d_k} ; and $|\varphi(p^{d_k}) - d_k| \leq |\varphi(p^{e_k}) - d_k|$, otherwise X could manipulate at p^{d_k} via p^{e_k} . Since $e_k \in (f^k \frown d_k)$, we obtain $\varphi(p^{e_k}) = \varphi(p^{d_k}) = f^k$ for every $k \in \mathbb{N}$. By similar arguments as before for the point a , it follows that e is on the perpendicular bisector L^{**} of $[f^* \frown \varphi(p^e)]$. Hence, $|e - \varphi(p^e)| = |e - f^*|$. But then

$$\begin{aligned} |a - \varphi(p)| &= |a - f^*| \text{ (since } a \in L^*) \\ &= |f^* - e| + |a - e| \text{ (since } e \in (f^* \frown a)) \\ &= |\varphi(p^e) - e| + |a - e| \text{ (since } e \in L^{**}) \\ &\geq |\varphi(p^e) - a| \text{ (by triangular inequality).} \end{aligned}$$

Hence, $|a - \varphi(p)| \geq |\varphi(p^e) - a|$, which by intermediate strict strategy-proofness implies $\varphi(p^e) = \varphi(p)$, and thus $e \in L^*$, contradicting $e \in (f^* \frown a)$. ■

We can now state our first main result.

Corollary 3.2. *Let φ be Pareto optimal and strict strategy-proof. Then φ is dictatorial.*

Proof. By Lemma 2.1 and Proposition 3.1, the set of decisive coalitions for φ is an ultrafilter. Hence φ is dictatorial. ■

We proceed by considering coalitional strategy-proofness.

Proposition 3.3. *Let φ be Pareto optimal and coalitional strategy-proof, and let S and T be decisive coalitions. Then $S \cap T$ is decisive.*

Proof. Contrary to what we wish to prove, suppose that $S \cap T$ is not decisive. Let X, Y , and Z be as in the proof of Proposition 3.1: as there, $X \cup Y, Y \cup Z$, and $X \cup Z$ are decisive. Consider a profile $p = (a^X, b^Y, c^Z)$ where $a, b, c \in A$ are distinct points in the same hemisphere and such that the angles in the triangle with edges $[a \frown b]$, $[b \frown c]$, and $[a \frown c]$ are less than 90° . By Remark 2.2, $\varphi(p) \in Co(\{a, b, c\})$.

Suppose $\varphi(p) \notin [a \frown b]$. Let m be the point of $[a \frown b]$ closest to $\varphi(p)$. Then $|a - m| < |a - \varphi(p)|$ and $|b - m| < |b - \varphi(p)|$, so that $X \cup Y$ can manipulate at p via $(m^{X \cup Y}, c^Z)$. This is a violation of coalitional strategy-proofness. Thus, $\varphi(p) \in [a \frown b]$. Similarly one shows that $\varphi(p) \in [c \frown a]$ and $\varphi(p) \in [b \frown c]$, but this is not possible. We conclude that $S \cap T$ is decisive. ■

By this proposition and Lemma 2.1 we obtain our second main result.

Corollary 3.4. *Let φ be Pareto optimal and coalitional strategy-proof. Then φ is dictatorial.*

It is an open problem whether Pareto optimality and strategy-proofness imply dictatorship. If not, coordinatewise median rules would be the typical candidate rules. The following example, however, suggests that such rules will be manipulable.

Example 3.5. In order to stay on the sphere while taking coordinatewise medians, we express the points of A by two coordinates, namely the angle with the vector $(1, 0, 0)$ and the angle with the vector $(0, 0, 1)$, both in degrees. Now suppose there are three agents with peaks $p(1) = (0^\circ, 0^\circ)$, $p(2) = (0^\circ, 90^\circ)$, and $p(3) = (90^\circ, 45^\circ)$. In Euclidian coordinates these are the points $(0, 0, 1)$, $(1, 0, 0)$, and $(0, \frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$, respectively. The coordinatewise median rule assigns the point $(0^\circ, 45^\circ)$, hence $(\frac{1}{2}\sqrt{2}, 0, \frac{1}{2}\sqrt{2})$ in Euclidian coordinates. If agent 3 reports $\tilde{p}(3) = (90^\circ, 0^\circ)$ instead, then the coordinatewise median rule would assign the point $(0^\circ, 0^\circ)$ or $(0, 0, 1)$, and it is easy to check that this is closer to agent 3's true peak $(0, \frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$ than the sincere outcome $(\frac{1}{2}\sqrt{2}, 0, \frac{1}{2}\sqrt{2})$ is. Thus, agent 3 manipulates successfully.

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